Indian Statistical Institute
B.Math. (Hons.) II Year
Second Semester 2006-07
Mid Semester Examination
Optimization

Time: 3 hrs | Date:06-03-07 | Max. Marks: 60

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This paper carries 70 marks. Maximum you can score is 60 marks. Answer as much as you can.

1. Consider the set $\{X : \ AX \leq b, \ X \geq 0\}$ where $A$ is $m \times n$ matrix and $b$ is an $m$-vector. Show that a nonzero vector $d$ is a direction of the set if and only if $Ad \leq 0$ and $d \geq 0$. [5]

2. (a) State Farka's theorem.

   (b) Let $A$ be an $m \times n$ matrix. Using Farka's theorem, prove that exactly one of the following two systems has a solution

$$\begin{array}{ll} \text{system 1} & Ax > 0 \\ \text{system 2} & A^t y = 0, \ y \geq 0 \ \ y \neq 0. \end{array}$$

[10]

3. Let $S = \{X : \ AX = b, \ X \geq 0\}$ be nonempty, where $A$ is an $m \times n$ matrix of rank $m$ and $b$ is an $m$-vector.

   (a) If $X = (X_1, X_2 \ldots X_l, O, O \ldots 0)^{1^t}$ is an extreme point of $S$, show that the corresponding column $a_1, a_2 \ldots a_l$ are linearly independent.

   (b) If for any solution $x$ with $p$ element positive such that $Ax = b, \ x \geq 0$ the corresponding column of $A$ are linearly dependent, that is, $x = (x_1, x_2 \ldots x_p, O, O, \ldots)$ and $\underset{\sim}{a_1}, \ \underset{\sim}{a_2}, \ \ldots, \underset{\sim}{a_p}$ as linearly dependent, then show that there exists an $x'$ with $(p-1)$ elements positive which solves $Ax = b, \ \ x \geq 0$. Hence conclude that whenever a feasible solution exists to an Linear Programming problem, there exists a basic feasible solution for the Linear Programming problem. [10]

4. Use Two-Phase Simplex Method to solve

   Maximize

$$Z = 5x_1 - 4x_2 + 3x_3$$

   subject to

$$\begin{aligned} 2x_1 + x_2 - 6x_3 &= 20 \\ 6x_1 + 5x_2 + 10x_3 &\leq 76 \\ 8x_1 - 3x_2 + 6x_3 &\leq 50 \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

[10]

5. Use *Big M-method* to solve

   Maximize $Z = 6x_1 + 4x_2$ subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 30 \\ 3x_1 + 2x_2 &\leq 24 \\ x_1 + x_2 &\geq 3 \\ x_1 \geq 0, \quad x_2 &\geq 0. \end{aligned}$$

   Is the solution unique? If not, find the two different solutions. [10]

6. Use dual Simplex Method to solve

   Minimize $Z = 10x_1 + 6x_2 + 2x_3$

   subject to

$$\begin{aligned} -x_1 + x_2 + x_3 &\geq 1 \\ 3x_1 + x_2 - x_3 &\geq 2 \\ x_1 \geq 0,\ x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

[10]

7. A company wants to produce three products A, B and C. These products require two types of resources. The profit per unit of A, B and C are Rs. 4, Rs. 6 and Rs. 2. The LP model is formulated as follows:

Maximize $Z = 4x_1 + 6x_2 + 2x_3$ subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 3 \qquad \text{(Man power constraint)} \\ x_1 + 4x_2 + 7x_3 &\leq 9 \qquad \text{(Raw material constraint)} \\ x_1 \geq 0, \quad x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

where $x_1, x_2, x_3$ are the number of units of A, B and C respectively.

(a) Find the optimal product mix and the corresponding profit for the company.

(b) Identify the *dual* and find its solution from the optimal tableau.

(c) Find the range of profit contribution of products A and C in the objective function such that current optimal product mix remains unchanged.

(d) If resource 1 is increased from 3 to $3 + \Delta b$, find the range for $\Delta b$, so that the present basis remains optimal. [15]